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CERTAIN CONTACT PROBLEMS OF THE THEORY OF ELASTICITY FOR AN ANNULAR SECTOR AND A SPHERICAL LAYER SECTOR*

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Two static contact problems of the theory of elasticity on the impression of a stamp in the circular boundary of an annular sector (Fig.1), and in the spherical surface of a spherical layer sector (Fig.2) are examined. By using homogeneous solutions the problems are reduced to an investigation of the well-studied integral equations that occur in the investigation of analogous problems, respectively, for a ring and a spherical layer, and infinite systems of linear high-quality algebraic equations of the type of the normal Poincaré-Koch systems.

A proof is also presented of the generalized orthogonality relationships (GOR) used for homogeneous solutions of the theory of elasticity on the steady vibrations of a spherical layer in the case of axial symmetry and a ring. In a special case, the GOR for a spherical layer agrees with those already known /1/, where the static problem is considered. Analogous GOR for a ring are proved by another method in /2, 3/, where the GOR are derived in /3/ as a corollary of the Betti reciprocity theorem for a broad class of media and domains.

The GOR are derived below as a corollary from values of a certain integral of the combination of two different solutions of the Lamé equation in the general case with arbitrary boundary conditions. The value of the integral is expressed in terms of boundary functions /4/. Values of the integral of both the homogeneous (generalized orthogonality condition), and the inhomogeneous solutions are used in deriving the infinite systems.

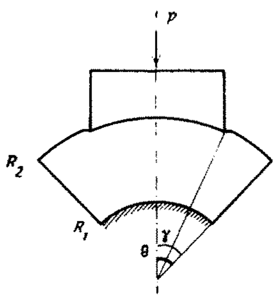

Fig.1

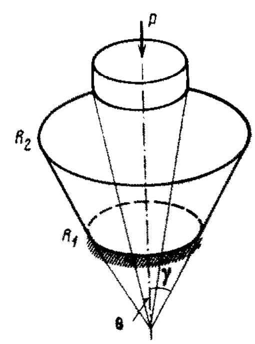

Fig.2

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1. In a spherical r, θ, φ coordinate system we consider the homogeneous solutions of the Lamé equations in axisymmetric problems of the steady vibrations of a spherical layer $R_1 \leqslant r \leqslant R_2$, whose edges $r = R_1$ and $r = R_2$ are: a) fixed, b) stress-free, or c) the edge $r = R_1$ is fixed while $r = R_2$ is stress free (or conversely). The eigenfunctions of these problems will be sought in the form

$$u_{rk}(r, \varphi) = W_k^\circ(r) P_{\alpha_k - 1/2}(\cos\varphi) e^{i\omega t}, \qquad (1.1)$$
$$u_\varphi\ (r, \varphi) = V_k^\circ(r) \frac{d}{d\varphi} P_{\alpha_k - 1/2}(\cos\varphi) e^{i\omega t}$$

where $u_{\varphi k}$, u_{rk} are projections of the displacement vector, respectively, on the φ and r axes, $P_{\alpha_k - 1/2}(\cos\varphi)$ are Legendre functions, ω is the frequency of vibration, t is the time, and α_k $(k = 1, 2, 3, \ldots)$ are eigenvalues. To determine $W_k^\circ(r)$ and $V_k^\circ(r)$ we obtain a system of ordinary differential equations

$$L_1(W_k^\circ, V_k^\circ) \equiv \mu r V_k^{\circ\prime\prime} + 2\mu V_k^{\circ\prime} - [(\lambda + 2\mu)(\alpha_k^2 - 1/4) r^{-1} - \rho\omega^2 r] V_k^\circ + (\lambda + \mu) W_k^{\circ\prime} + 2(\lambda + 2\mu) r^{-1} W_k^\circ = 0 \qquad (1.2)$$

$$L_2(W_k^\circ, V_k^\circ) \equiv (\lambda + 2\mu) r W_k^{\circ\prime\prime} + 2(\lambda + 2\mu) W_k^{\circ\prime} - [2(\lambda + 2\mu) + (\alpha_k^2 - 1/4)\mu - \rho\omega^2 r^2] r^{-1} W_k^\circ - (\lambda + \mu)(\alpha_k^2 - 1/4) V_k^{\circ\prime} + (\lambda + 3\mu)(\alpha_k^2 - 1/4) r^{-1} V_k^\circ = 0 \qquad (1.3)$$

where λ, μ are the Lamé coefficients, ρ is the density, and L_1 and L_2 are appropriate differential operators.

The boundary conditions take the form

$$W_k^\circ(R_1) = W_k^\circ(R_2) = V_k^\circ(R_1) = V_k^\circ(R_2) = 0 \qquad (1.4)$$
$$\sigma_{rk}^*(R_1) = \sigma_{rk}^*(R_2) = \tau_k^*(R_1) = \tau_k^*(R_2) = 0$$
$$W_k^\circ(R_1) = V_k^\circ(R_1) = \sigma_{rk}^*(R_2) = \tau_k^*(R_2) = 0$$

The stress tensor components here are

$$\sigma_{rk} = \sigma_{rk}^*(r) P_{\alpha_k - 1/2}(\cos\varphi) e^{i\omega t} \qquad (1.5)$$
$$\tau_{r\varphi k} = \tau_k^*(r) \frac{d}{d\varphi} P_{\alpha_k - 1/2}(\cos\varphi) e^{i\omega t}$$
$$\sigma_{\varphi k} = \Big[\sigma_{\varphi k}^*(r) P_{\alpha_k - 1/2}(\cos\varphi) - 2\mu r^{-1} V_k^\circ(r) \operatorname{ctg}\varphi \times \frac{d}{d\varphi} P_{\alpha_k - 1/2}(\cos\varphi)\Big] e^{i\omega t}$$
$$\sigma_{rk}^* = 2\lambda r^{-1} W_k^\circ(r) + (\lambda + 2\mu) W_k^{\circ\prime} - \lambda(\alpha_k^2 - 1/4) r^{-1} V_k^\circ$$
$$\tau_k^* = \mu(-r^{-1} V_k^\circ + V_k^{\circ\prime} + r^{-1} W_k^\circ)$$
$$\sigma_{\varphi k}^* = 2(\lambda + \mu) r^{-1} W_k^\circ + \lambda W_k^{\circ\prime} - (\lambda + 2\mu)(\alpha_k^2 - 1/4) r^{-1} V_k^\circ$$

Theorem. If problem (1.2)-(1.4) has just a simple eigenvalues and $\alpha_k^2 \neq \alpha_n^2$ then the following GOR hold:

$$\mathring{W}_{kn} \equiv \int_{R_1}^{R_2} [V_k^\circ(r) \sigma_{\varphi n}^*(r) - \tau_k^*(r) W_n^\circ(r)] r\, dr = 0 \qquad (1.6)$$

Proof. Using (1.2) and (1.3), we consider the obvious equalities

$$\int_{R_1}^{R_2} [V_n^\circ L_2(W_k^\circ, V_k^\circ) - V_k^\circ L_2(W_n^\circ, V_n^\circ)] r\, dr = 0$$
$$\int_{R_1}^{R_2} [W_n^\circ L_1(W_k^\circ, V_k^\circ) - W_k^\circ L_1(W_n^\circ, V_n^\circ)] r\, dr = 0$$

from which we find, respectively

$$-(V_k^\circ V_n^\circ)(\lambda + 2\mu)(\alpha_k^2 - \alpha_n^2) = \mu [r^2 V_k^\circ V_n^{\circ\prime} - r^2 V_n^\circ V_k^{\circ\prime}]_{R_1}^{R_2} + (\lambda + \mu)[(r V_k^\circ W_n^{\circ\prime}) - (r V_n^\circ W_k^{\circ\prime})] + 2(\lambda + 2\mu)[(V_k^\circ W_n^\circ) - (V_n^\circ W_k^\circ)]; \quad -(W_k^\circ W_n^\circ)\mu(\alpha_k^2 - \alpha_n^2) = (\lambda + 2\mu)[r^2 W_k^\circ W_n^{\circ\prime} - r^2 W_n^\circ W_k^{\circ\prime}]_{R_1}^{R_2} - (\lambda + \mu)(\alpha_n^2 - 1/4)(r V_n^{\circ\prime} W_k^\circ) + (\lambda + \mu)(\alpha_k^2 - 1/4)(r V_k^{\circ\prime} W_n^\circ) + (\lambda + 3\mu)(\alpha_n^2 - 1/4)(V_n^\circ W_k^\circ) - (\lambda + 3\mu)(\alpha_k^2 - 1/4)(V_k^\circ W_n^\circ) \qquad (1.7)$$

$$(r^m/g) \equiv \int_{R_1}^{R_2} r^m f(r) g(r)\, dr \quad (m = 0, 1)$$

Furthermore, we consider the expression

$$(\alpha_k^2 - \alpha_n^2) W_{kn}^\circ \tag{1.8}$$

where W_{kn}° is given by relationship (1.6). We replace $\sigma_{\varphi n}{}^*$ and $\tau_k{}^*$ in (1.8) by expressions using formulas from (1.5) and we expand the parentheses, after which we replace corresponding components in the relationships newly obtained by expressions on the right sides of (1.7). Integrating by parts, we evaluate the integral (1.8) and we add and subtract the expression

$$[2\lambda r W_n{}^\circ W_k{}^\circ + \lambda (\alpha_k^2 - {}^1/_4) r W_n{}^\circ V_k{}^\circ + \mu (\alpha_n^2 - {}^1/_4) r V_k{}^\circ V_n{}^\circ]_{R_1}^{R_2}$$

to the relationship obtained.

Regrouping the components in an appropriate manner, we finally obtain

$$\begin{aligned} \overset{\circ}{W}_{kn} = (\alpha_k^2 - \alpha_n^2)^{-1} [r^2 W_k{}^\circ \sigma_{rn}^* - r^2 W_n{}^\circ \sigma_{rk}^* + \\ r^2 V_k{}^\circ (\alpha_n^2 - {}^1/_4) \tau_n{}^* - r^2 V_n{}^\circ (\alpha_n^2 - {}^1/_4) \tau_k{}^*]_{R_1}^{R_2} \end{aligned} \tag{1.9}$$

It follows from (1.9) that $W_{kn}{}^\circ = 0$ for any boundary conditions (1.4) if $\alpha_n^2 \neq \alpha_k^2$.

The derivation of (1.9) is not related to the form of the boundary conditions for the Lamé equations, therefore, they will be valid for any inhomogeneous boundary conditions. In this case, $\alpha_k = k$ should be considered in the particular inhomogeneous solutions of the form (1.1) and formulas (1.9). The case when one of the solutions in (1.9) is inhomogeneous while the other is homogeneous is needed to evaluate integral (3.12).

As is seen from the proof, the GOR (1.6) hold even for $\omega = 0$. In this case they agree with what is known /1/.

2. In a cylindrical r, φ, z coordinate system we consider the homogeneous solution of the Lamé equations in plane problems on the steady vibrations of a ring $R_1 \leqslant r \leqslant R_2$, on whose edges $r = R_1$ and $r = R_2$ arbitrary homogeneous conditions are given, analogous to the conditions for the problems of Sect.1. If the eigenfunctions of such problems are written in the form

$$u_{rk} = W_k{}^\circ(r) \cos \alpha_k \varphi e^{i\omega t}, \quad u_{\varphi k} = V_k{}^\circ(r) \sin \alpha_k \varphi e^{i\omega t} \tag{2.1}$$

where α_k $(k = 1, 2, \ldots)$ are the eigenvalues, then the corresponding stress tensor components will take the form

$$\begin{aligned} \sigma_{qk} &= E_* \sigma_{qk}{}^\circ(r) \cos \alpha_k \varphi\, e^{i\omega t}, \; q = r, \varphi \\ \tau_{r\varphi k} &= E_* \tau_k{}^\circ(r) \sin \alpha_k \varphi\, e^{i\omega t}; \; E_* = E/(1 - \nu^2) \end{aligned} \tag{2.2}$$

As in Sect.1, by using the system of equations for $W_k{}^\circ$ and $V_k{}^\circ$ we can evaluate the integral

$$\begin{aligned} \overset{\circ}{W}_{kn} \equiv \int_{R_1}^{R_2} [\overset{\circ}{\sigma}_{\varphi k}(r) V_n{}^\circ(r) - \tau_n{}^\circ(r) W_k{}^\circ(r)]\, dr = \\ r (\alpha_k^2 - \alpha_n^2)^{-1} [\alpha_k V_n{}^\circ \tau_k{}^\circ - \alpha_k V_k{}^\circ \tau_n{}^\circ + \alpha_n W_n{}^\circ \overset{\circ}{\sigma}_{rk} - \alpha_n W_k{}^\circ \overset{\circ}{\sigma}_{rn}]_{R_1}^{R_2} \end{aligned} \tag{2.3}$$

It follows from this last relationship that for any homogeneous boundary conditions of the type (1.4) we obtain the GOR

$$W_{kn}{}^\circ = 0 \tag{2.4}$$

if $\alpha_k^2 \neq \alpha_n^2$ and all the α_k are simple eigenvalues.

We note that, as in (1.9), the value of the integral (2.3) is independent of the form of the boundary conditions, and the case when one of the particular solutions is inhomogeneous will also be utilized later (Sect.4).

The relationships (2.4) are also valid for $\omega = 0$ and agree with those already known /2, 3/.

The GOR (1.6) and (2.4) afford the possibility of an effective investigation of a broad class of axisymmetric problems for the sector of a spherical layer and plane problems for an annular sector by using homogeneous solutions. The method described in /5, 6/, say, can be used to solve such contact problems when mixed boundary conditions are given on spherical and cylindrical surfaces.

3. We consider the static $(\omega = 0)$ contact problem for an annular sector $R_1 \leqslant r \leqslant R_2$, $|\varphi| \leqslant \gamma$ for the impression of a stamp in the face $r = R_2$, let the face $r = R_1$ here lie without friction on a rigid foundation, and let there be no tangential stresses and normal displacements (Fig.1) on the faces $\varphi = \pm\gamma$.

The boundary conditions of such a problem are (δ is the stamp displacement)

$$u_r = \delta \cos \varphi \, (r = R_2, \; |\varphi| \leqslant \theta) \tag{3.1}$$

$$\begin{aligned}
&\sigma_r = 0 \ (r = R_2,\ \theta < |\varphi| < \gamma) \\
&\tau_{r\varphi} = 0 \ (r = R_1,\ r = R_2) \\
&u_r = 0 \ (r = R_1) \\
&\tau_{r\varphi} = 0,\ u_\varphi = 0 \ (|\varphi| \leqslant \gamma)
\end{aligned}$$

Under conditions (3.1) we will seek the solution of the Lamé equations in the form

$$u_r(r,\varphi) = u_r^{(1)} - u_r^{(2)}, \quad u_\varphi(r,\varphi) = u_\varphi^{(1)} - u_\varphi^{(2)} \tag{3.2}$$

where $u_r^{(1)}$, $u_\varphi^{(1)}$ are the solution of the Lamé equations for the ring when the following boundary conditions are given:

$$\begin{aligned}
&\sigma_r(R_2,\ \varphi) = \{q(r), \ \text{if } |\varphi| \leqslant \theta;\ 0, \ \text{if } |\varphi| > \theta\} \\
&\tau_{r\varphi}(R_2,\ \varphi) = 0,\ \tau_{r\varphi}(R_1,\ \varphi) = u_r(R_1,\ \varphi) = 0
\end{aligned} \tag{3.3}$$

where $u_r^{(2)}$, $u_\varphi^{(2)}$ are the superposition of homogeneous solutions of the Lamé equations for a ring when the following boundary conditions are given:

$$\sigma_r(R_2,\varphi) = \tau_{r\varphi}(R_2,\varphi) = 0,\ \tau_{r\varphi}(R_1,\varphi) = u_r(R_2,\varphi) = 0$$

In this case the solutions of the Lamé equations with boundary conditions (3.3) have the form

$$\begin{aligned}
&u_r^{(1)} = \frac{1}{E_*}\sum_{k=0}^{\infty}\frac{a_k}{\Delta(k)}W_k(r)\cos k\varphi \\
&u_\varphi^{(1)} = \frac{1}{E_*}\sum_{k=0}^{\infty}\frac{a_k}{\Delta(k)}V_k(r)\sin k\varphi
\end{aligned} \tag{3.4}$$

while the corresponding stress tensor components are

$$\begin{aligned}
&\sigma_q^{(1)} = \sum_{k=0}^{\infty}\frac{a_k}{\Delta(k)}\sigma_{qk}(r)\cos k\varphi, \quad q = r, \varphi \\
&\tau_{r\varphi}^{(1)} = \sum_{k=1}^{\infty}\frac{a_k}{\Delta(k)}\tau_k(r)\sin k\varphi; \quad a_k = \frac{2}{\pi}\int_0^\theta q(t)\cos kt\,dt
\end{aligned} \tag{3.5}$$

The functions $W_k(r)$, $V_k(r)$, $\sigma_{rk}(r)$, $\tau_k(r)$ and $\sigma_{\varphi k}(r)$ in (3.4) and (3.5) are known (later only $W_k(r)$ and $\Delta(k)$ are needed); $q(t)$ is the contact pressure distribution function that must be determined. We have

$$\begin{aligned}
&u_r^{(2)} = \sum D_k W_k^\circ(r)\cos\alpha_k\varphi, \quad u_\varphi^{(2)} = \sum D_k V_k^\circ(r)\sin\alpha_k\varphi \\
&\sigma_q^{(2)} = E_*\sum D_k \overset{\circ}{\sigma}_{qk}(r)\cos\alpha_k\varphi, \quad q = r, \varphi \\
&\tau_{r\varphi}^{(2)} = E_*\sum D_k\tau_k^\circ(r)\sin\alpha_k\varphi
\end{aligned} \tag{3.6}$$

Summation here is over all the zeros α_k of the function $\Delta(\alpha_k)$ in the right half-plane. Note that $\Delta(k) = \Delta^\circ(k)$ $(k > 2)$.

Therefore, the functions (3.2), (3.4), (3.6) satisfy the Lamé equations and the boundary conditions (3.1), except the first and last, which now take the form

$$\begin{aligned}
&u_r^{(1)}(R_2,\varphi) - u_r^{(2)}(R_2,\varphi) = \delta\cos\varphi \quad (|\varphi| \leqslant \theta) \\
&u_\varphi^{(1)}(r,\gamma) - u_\varphi^{(2)}(r,\gamma) = 0, \quad \tau_{r\varphi}^{(1)}(r,\gamma) - \tau_{r\varphi}^{(2)}(r,\gamma) = 0 \quad (R_1 \leqslant r \leqslant R_2)
\end{aligned} \tag{3.7}$$

We will represent the unknown contact stresses in the form

$$q(t) = E_*\left[\delta q_0(t) + \sum_{k=1}^{\infty} D_k W_k^\circ(R_2)\, q_k(t)\right] \tag{3.8}$$

By satisfying the first boundary conditions in (3.7), we obtain a number of integral equations to determine $q_k(t)$ $(k = 0, 1, 2, \ldots)$

$$K_\varphi q_0 = \cos\varphi;\ K_\varphi q_k = \cos\alpha_k\varphi,\ k \geqslant 1 \ (|\varphi| \leqslant \theta) \tag{3.9}$$

where the integral operator K_φ can be reduced to the form

$$K_\varphi q \equiv \int_{-\theta}^{\theta} M(t-\varphi)\,q(t)\,dt, \quad M(y) = \frac{1}{\pi}\sum_{k=3}^{\infty}\frac{W_k(R_2)}{\Delta(k)}\cos ky \tag{3.10}$$

when the evenness of the function $q(t)$ is taken into account.

We rewrite the last two conditions of (3.7) in the form

$$\sum_{k=1}^{\infty} D_k V_k^{\circ}(r)\sin\alpha_k\gamma = u_{\varphi}^{(1)}(r,\gamma)$$

$$\sum_{k=1}^{\infty} D_k \tau_k^{\circ}(r)\sin\alpha_k\gamma = \frac{1}{E_*}\tau_{r\varphi}^{(1)}(r,\gamma)$$

We multiply the first relationship by $\sigma_{\varphi n}^{\circ}$ and subtract the second relation multiplied by W_n° from the equality obtained; we then integrate the exression obtained within the limits R_1 and R_2. Using the GOR (2.4) we obtain

$$D_k W_{kk}^{\circ}\sin\alpha_k\gamma = \frac{1}{E_*}\sum_{n=1}^{\infty}\frac{a_n\sin n\gamma}{\Delta(n)}\int_{R_1}^{R_2}[V_n\sigma_{\varphi k}^{\circ} - \tau_n W_k^{\circ}]\,dr \tag{3.11}$$

Taking into account that the pairs of functions V_n, W_n and V_n°, W_n° satisfy the identical system of equations, the integral from (3.11) can be evaluated by using the integral (2.3) and boundary conditions (3.3) and (3.4)

$$\int_{R_1}^{R_2}[V_n\sigma_{\varphi k}^{\circ} - \tau_n W_k^{\circ}]\,dr = -\frac{nR_2W_k^{\circ}(R_2)\sigma_{rn}(R_2)}{\alpha_k^2 - n^2} \tag{3.12}$$

Since

$$\sigma_{rn}(R_2) = \Delta(n), \qquad \sum_{n=1}^{\infty}\frac{n\cos nt\sin nt}{n^2-\alpha_k^2} = \frac{\pi}{2}\,\frac{\sin(\pi-\gamma)\alpha_k\cos t\alpha_k}{\sin\pi\alpha_k}$$

expression (3.11) can be converted to the form

$$D_k W_{kk}^{\circ}\sin\alpha_k\gamma = \frac{R_2W_k^{\circ}(R_2)\sin(\pi-\gamma)\alpha_k}{E_*\sin\pi\alpha_k}\int_0^{\theta} q(t)\cos\alpha_k t\,dt \tag{3.13}$$

Substituting (3.8) into (3.13), we obtain an infinite system to determine the coefficients D_k

$$y_k = b_k + \sum_{k=1}^{\infty} a_{kn}y_n \quad (k=1,2,\ldots), \quad D_k = \frac{y_k}{W_k^{\circ}(R_2)\sin\alpha_k\gamma} \tag{3.14}$$

$$b_k = \delta c_k T_{k0}, \; a_{kn} = c_k T_{kn}/\sin\alpha_n\gamma$$

$$c_k = \frac{R_2W_k^{\circ 2}(R_2)\sin(\pi-\gamma)\alpha_k}{W_{kk}^{\circ}\sin\pi\alpha_k}, \quad T_{kn} = \int_0^{\theta} q_n(t)\cos\alpha_k t\,dt$$

It can be shown that $T_{kn} = T_{nk}$.

Therefore, formula (3.8) is obtained to seek the contact stress distribution function under the stamp, where $q_k(t)$ are found from the integral Eqs. (3.9) and (3.10) while D_k are found from the infinite system (3.14). We note that analogous contact problems for a circular ring reduce to (3.9) and (3.10). Therefore, the problem of solving the contact problem for an annular sector is here reduced to the contact problem for a ring already well studied. Moreover, the infinite system (3.14) refers to the type of normal Poincaré-Koch systems, i.e., its coefficients b_k and a_{kn} decrease as the numbers in the exponential grow, as will be shown below. Therefore, its solution can be obtained by the method of reduction for any values of the parameters.

As is seen from (3.8)-(3.10) and (3.14), only the expressions $W_k(R_2)\,\Delta^{-1}(k)$, $W_k^{\circ}(R_2)$, W_{kk}° and the eigennumbers α_k are needed to investigate the solution of the problem. It is easy to obtain the mentioned expressions (see /7/, say) and they are not presented here; we merely find the asymptotic form of the numbers α_k for large k. As has been noted, α_k are the roots of the equation

$$\Delta^{\circ}(\alpha_k) \equiv 2(1-\nu)\,\mathrm{ch}\,(2\alpha_k\ln\varkappa) + 4\alpha_k\mathrm{sh}\,(2\alpha_k\ln\varkappa) + \mu_0\alpha_k^2 - 2(1-\nu) = 0, \; \varkappa = R_2/R_1 \tag{3.15}$$

$$\mu_0 = \varkappa^2(1+\nu) - \varkappa^{-2}(3-\nu) + 2(1-\nu)$$

It can be established that for large numbers the roots of this equation have the following asymptotic form:

$$\alpha_k \sim [\ln (|\pi\mu_0 (2\ln \varkappa)^{-1}| (k - {}^1/_4)) + i\, 2\pi (k - {}^1/_4)] (2 \ln \varkappa)^{-1} \tag{3.16}$$

which, in turn, enables us to estimate the matrix elements on the right-hand side of the infinite system (3.14) for large k, n

$$| b_k | \leqslant P_k \exp [-\beta_k (\gamma - \theta)], \quad | a_{kn} | \leqslant P_{kn} \exp [-(\beta_k + \beta_n)(\gamma - \theta)]$$
$$\beta_k = \pi (k - {}^1/_4)/\ln \varkappa$$

where P_k and P_{kn} are bounded quantities. This confirms the fact that the infinite system (3.14) refers to the type of normal Poincaré-Koch systems.

Remark. 1°. The relationship $d_k = o\,[\exp(-\beta_k(\gamma - \theta))]$ is satisfied for the elements $d_k = D_k W_k^\circ (R_2)\, q_k(t)$ of the series (3.8) for any $|t| < \theta$ and large k, and therefore, for $|t| < \theta$ the series (3.8) converges not more slowly than the sum of terms of a geometric progression with denominator less than one.

2°. Taking account of the asymptotic form of the function $W_k(R_2)/\Delta(k)$ for large k, the kernel of the integral Eqs.(3.9), (3.10) can be represented in the form

$$M(y) = -\frac{2R_2}{\pi(1-\nu^2)} \ln \left| 2 \sin \frac{y}{2} \right| + F_1(y) \tag{3.18}$$

or

$$M(y) = \frac{2R_2}{\pi(1-\nu^2)} \sum_{k=0}^{\infty} \frac{\operatorname{cth}(2k \ln \varkappa)}{k} \cos ky + F_2(y) \tag{3.19}$$

where $F_1(y)$ and $F_2(y)$ are continuous functions for all $|y| < M < \infty$. The representation of the kernel in the form (3.18) enables an effective solution to be obtained for the integral Eqs.(3.9), (3.10) by the method of orthogonal polynomials /8/, while the representation (3.19) permits exact inversion /9/ of the principal part of the integral operator of (3.9) and (3.10) and their reduction to integral equations of the second kind.

4. As in Sect.3, the same contact problem can be considered on the impression of a stamp in the spherical surface of a spherical layer sector in the case of axial symmetry (Fig.2). In this problem the contact stresses under the stamp are

$$q(t) = \delta q_0(t) + \sum_{n=1}^{\infty} D_n W_n^\circ (R_2)\, q_n(t) \tag{4.1}$$

Here δ is the stamp displacement, $q_n(t)$ $(n = 0, 1, 2, \ldots)$ are determined from the integral equations

$$K_\varphi q_0 = \cos\varphi, \quad K_\varphi q_k = P_{\alpha_k - 1/2}(\cos\varphi) \quad (|\varphi| \leqslant \theta) \tag{4.2}$$
$$K_\varphi q \equiv \int_0^\theta q(t) \sin t\, dt \sum_{k=0}^{\infty} \frac{W_k(R_2)}{\Delta(k)} \frac{2k+1}{2} P_k(\cos t)\, P_k(\cos\varphi)$$

and the D_n are found from the infinite system

$$x_k = b_k + \sum_{n=1}^{\infty} a_{kn} x_n \tag{4.3}$$
$$D_k = \frac{x_k}{W_k^\circ(R_2) \cos \alpha_k \gamma}, \quad b_k = -\delta c_k T_{0k}$$
$$a_{kn} = -c_k T_{nk} / \cos \alpha_n \gamma$$
$$c_k = \frac{\pi R_2^2 W_k^{\circ 2}(R_2) \cos \alpha_k \gamma}{2 W_{kk}^\circ \cos \pi \alpha_k}$$
$$T_{nk} = \int_0^\theta q_n(t)\, P_{\alpha_k - 1/2}(\cos t) \sin t\, dt, \quad T_{kn} = T_{nk}$$

The notation in (4.1)-(4.3) is taken by analogy with the preceding problem, $W_k^0(R_2)$ corresponds to the homogeneous problem, $W(R_2)/\Delta(k)$ to the inhomogeneous problem, and α_k to the roots of the equation $\Delta(\alpha_k)$ in the right half-plane. Without investigating series (4.1), the integral Eqs.(4.2), and the infinite system (4.3), we note that here, as in the problem of Sect.3, it can be shown that the elements b_k and a_{kn} of system (4.3) decreases as the numbers increase in the exponential, the series in (4.1) converges no more slowly than the sum of the terms of an infinitely decreasing geometric progression, while the solution of the integral Eqs.(4.2) can be obtained by using a large set of effective methods including the asymptotic methods developed for a similar class of equations (/10, 11/, for instance).

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DYNAMIC PROPERTIES OF AN ELASTIC SEMIBOUNDED MEDIUM IN THE PRESENCE OF TWO MASSIVE STAMPS*

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The dynamic properties of a system consisting of two massive rigid strip stamps and an elastic semi-infinite medium are investigated. A layer, a cylinder, a multilayer foundation, etc., can be selected as such a medium. The method of fictitious absorption is used, which are developed for one stamp in /1/. Unlike other approaches to solve there problems /2-4/, this method enables one to describe, to any degree of accuracy, the behaviour of contact stresses simultaneously at all points of the contact domain, both inside and on the boundary.

The presence of resonance frequencies of four kinds is established in the system. Among the first kind is the value of the frequency $\varkappa_{2*}$, starting with which the system has no energetic solution and waves propagate therein that have only geometric damping. The critical frequency here is independent of the stamp characteristics and is determined just by the geometric and dynamic properties of the waveguide. The second kind of resonances is characterized by the frequencies to which multiple roots correspond, i.e., the poles of the

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